Fractional partial Differential Equations for Laplace transformation Caputo-Fabrizo and Volterra integration

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Abstract:

In this paper we note the numerical methods for solving fractional differential equations, defined in the derivative of the Caputo-Fabrizio fractional operator and Laplace transform of fractional derivatives for integer order, solving differential equation problems using the Laplace transform method, and reducing to Volterra's integral equation, Laplace transform of the Mittage–Leffler function, this problem is not easy to solve analytically because an analytical solution is sometimes not available, even if an analytical solution is available, but it is complected, time-consuming and expensive, so we need to develop a numerical method to address the relevant problem, Analyze a precise result such as the integral or exact expression of a solution to obtain a qualitative answer that shows us what is happening with each variable while numerical methods are more adaptable in the approximate result to obtain quantitative results by iteratively creating an approximate solution sequence for mathematical problems. The method will solve a non-homogeneous linear differential equation directly, following basic steps, without having to solve the integral equation and solutions separately and non-linear differential equations with the rational factor by developing analytical or numerical techniques to find approximate solutions. Finally, we studied some applications, especially for nonlinear differential equations with the rational operator.

Keywords: Laplacian Transform Interpretation, fractional Caputo–Fabrizio derivative operator, the Volterra Integral Equation, Existence and uniqueness, Iterative Laplace transform method.

1. Introduction

The mathematical models involving fractional derivatives were given noticeable importance because they are more accurate and realistic as compared to the classical order models [1,2]. Fractional differential equation, particularly fractional calculus equation and derivatives of functions gamma function while investigating the interpolation problem. There are several approaches leading to the definition of gamma function. However, this in Mathematics we are not looking at the usual integer order but at the non-integer order differential, and derivatives [3]. The Riemann–Liouville fractional differential operators have played a significant role in the development of the theory of differentiation and integration of arbitrary order, the Method of Volterra Integral Equation, Laplace transform of the Mittage–Leffler function, we introduced the series which converges to the solution of an initial-value to Volterra integral problem [4]. These are called fractional derivatives and fractional integrals, which can be of real or complex integer, and therefore also include integer orders. In this study, we refer if we are talking about the combination of these fractional derivatives has significant applications. Motivated by the advancement of fractional calculus, These differential equations involve several fractional differential operators like Riemann–Liouville, Caputo, [5], and modeling of materials and diffusion and expansion processes [6]. To avoid these problems, we find that the fractional partial differential operator has a substitution kernel with exponential decay [7]. Operator is best suited for modeling some classes as follows:

$$\begin{cases} D_{t_0}^{\alpha} y(t) \\ \\ y(t_0) \end{cases} \tag{1}$$

To confirm the existence and uniqueness of the solution to problem (1) suppose that $f(x, y)$. The function is continuous and fulfills Lipschitz's condition with respect to the second variable [8]. The initial value problem (1) can be transformed into an equivalent Volterra integral equation.

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-x)^{\alpha-1} f\big(x, y(x)\big) dx \tag{2}$$

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2. Definition of fractional calculus

We reviewed some definitions of the fractional partial derivative and the fractional integral. One should note that trigonometric functions of order n are generalizations of the sine and cosine functions of fractional calculus.

Definition 2.1 Fractional calculus is used for integrals and fractional partial derivatives [9]. It can be said that the order of numbers is truly arbitrary or even the order of a complex number. There are many definitions of the partial and integral derivation, such as we described. Other definitions can be found in [10,11]. Here we use D and I to denote the fractional derivative and the fractional integral, respectively.

Definition 2.2 It can be generalized that the integer-order classical partial derivation, which is used for continuous function $f(t)$ is.

$$f^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^2} \sum_{r=0}^{n} (-1)^i \binom{n}{i} f(t - ih), \tag{4}$$

Where $\binom{n}{r}$ is the binomial coefficients. If n is replaced by $\alpha \in \mathbb{R}$ we get

$$D^{\alpha}_{\alpha,t} f(t) = \lim_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{t-\alpha}{h}} (-1)^i \binom{\alpha}{i} f(t - ih), \tag{5}$$

where we denote the base function and the α denotes the starting point of the interval.

Definition 2.3 The Grünwald-Letnikov integral of arbitrary order is:

$$I^{\alpha}_{\alpha,t} f(t) \lim_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{t-\alpha}{h}} (-1)^i \binom{\alpha}{i} f(t - ih), \tag{6}$$

Definition 2.4 Riemann-Liouville. The αth order Riemann-Liouville derivative of function is.

$$D^{\alpha}_{\alpha,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_{\alpha}^{t} \frac{f(x)\, dx}{(t-x)^{1-\alpha}}, \tag{7}$$

And the integral

$$I^{\alpha}_{\alpha,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} \frac{f(x)\, dx}{(t-x)^{1-\alpha}} \tag{8}$$

Definition 2.5 The Riemann-Liouville definition is important for the development of fractional derivatives, but it is difficult to calculate the integral with physically explicable elementary points. [12] Caputo solved this issue by creating a new definition.

$$D_{\alpha,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{\alpha}^{t} (t-x)^{n-\alpha-1} \frac{\partial^n f(x)}{\partial x^n} \, dx, \;\; n-1 < \alpha < n \quad (9)$$

3. Numerical and Analytical methods

There are different ways to solve fractional differential equations analytically. One of the most popular and widely used methods is the Laplace transformation. Below, for example, this method is described [13]. Before continuing, it should be noted that in general, the number of initial conditions required for a partial differential equation will depend on the order of the differential equation. However, in a fractional differential equation, the number of the initial condition is equal to the minimum integer order value α [14,15]. Consider the following differential equation.

$$xD_t^{\alpha} y(t) + Ky(t) = f(t) \quad (10)$$

Which $y(t)$ is displacement, k, and τ are constants, as well as the fractional derivative is also Caputo and $0 < \alpha < 1$. In what follows, it has been shown that this partial differential equation model is the dynamics of a purely elastic spring and a viscoelastic element connecting in parallel with a body of mass m, which a force f is applied on a body.[9] To solve, the first step is to take the Laplace transformation of both sides of the original partial differential equation, the Laplace transformation is concisely explained. we have:

$$Y(s) = \frac{f(t)}{x(s^{\alpha} + k/x)} \quad (11)$$

where α and s are fractional order and Laplace domain variable respectively. Also, it is supposed that $x(0) = 0$. To find the solution, all we need to do is to take the inverse transform:

$$y(t) = \frac{f(t)}{x} t^{(\alpha-1)} E_{\alpha,\alpha}(t) \left(-\frac{k}{x} t^{\alpha}\right) \quad (12)$$

Which $E_{\alpha,\alpha}$ (t) is Mittag-Leffler function, if the spring is ignored, the equation (10) will be reduced to

$$f(t) = xD_t^{\alpha}y(t) \tag{13}$$

We taking the Laplace transforms of both sides of the equation, simplifying algebraically the result to solve the obtained equation in terms of s, and c finally finding the inverse transform, we have:

$$y(t) = Kt^{\alpha} \tag{14}$$

Which $K = f/x\Gamma(\alpha+1)$. Although the Laplace transformation method is one of the simple and practical methods for solving the fractional equations same as the partial differential equations, most of the fractional equations could not be solved analytically. In what follows, we present a numerical technique to solve Caputo fractional differential equation. So numerical simulations of fractional differential equations need a larger number of floating-point operations and data flow in computer memory systems. This is because, as pointed out by [16], specific additional conditions are needed to solve a differential equation to obtain a unique solution. These additional conditions for the Riemann–Liouville fractional derivative constitute a certain fractional derivative of unknown solution at the initial points which might result in an unclear physical meaning. Due to this reason, in the present work, we consider the fractional Caputo's.

$$y_k^p(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} b_{j,k}\, f\left(t_j, y_j\right), \tag{15}$$

4. Laplacian Transform Interpretation

Suppose that $Y(t)$ is a quantity whose value in terms of $f(t,y)$ can be achieved as follows:

$$Y(t) = \int_0^t \frac{(t-x)^{(\alpha-1)}}{\Gamma(\alpha)} f(x)dx \tag{16}$$

The output $Y(t)$ can be viewed as a power-weighted sum which stores the previous input of function $f(x)$. Based on the above definition, such system is a non-memoryless system and, in such systems, memory decays at the rate of $y_k^p(t) = t^{\alpha} - \frac{1}{\Gamma(\alpha)}$.

Applying the Caputo derivative of order α to both sides of the last relation led to

$$D_t^{\alpha}Y(t) = f(t) \tag{17}$$

As a result, the differential equation governing the system memory $Y(t)$ is described by a fractional derivative [17]. Therefore, the fractional derivative is a good candidate to explain the system with memory. The nature of weighted function determines the type of fractional derivative which describes a system memory. For example, If the weight function of a system is defined by $t^{1-\alpha} / \Gamma(\alpha)$, the Riemann Liouville elements, and by $t^{1-\alpha}\, \theta(x-t) / \Gamma(\alpha)$ the Caputo elements are used which θ is the Heaviside function [18].

Definition 4.1 Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}(f(t))$, is an integral transform given by the Laplace integral:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{18}$$

Theorem 4.1 The Laplace transform of the Mittage–Leffler function is given by the equation [19]:

$$\mathcal{L}^{-1}\left(\frac{s^{-(\gamma-\alpha)}}{s^\alpha - y}\right) = t^{\gamma-1} G_{\alpha,\gamma}(yt^\alpha), \quad |s^\alpha - y| < 1$$

Proof. Using the definition of the Laplace transform, we have:

$$\mathcal{L}\left(t^{\gamma-1} G_{\alpha,\gamma}(yt^\alpha)\right) = \int_0^{+\infty} e^{-st} t^\gamma G_{\alpha,\gamma}(yt^\alpha) dt = \sum_{i=1}^{+\infty} \frac{y^i}{\Gamma(\alpha i + \gamma)} = \int_0^{+\infty} e^{-st}\, t^{\alpha i + \gamma - 1}\, dt \tag{19}$$

From this equation we get

$$\sum_{i=0}^{+\infty} \frac{y^i}{\Gamma(\alpha i + \gamma)} \mathcal{L}\left(t^{\alpha i + \gamma - 1}\right) = \sum_{i=0}^{+\infty} \frac{y^i}{\Gamma(\alpha i + \gamma)} \frac{\Gamma(\alpha i + \gamma)}{S^{\alpha i + \gamma}} = \frac{1}{S^\gamma} \sum_{i=0}^{+\infty} \left(\frac{y}{S^\alpha}\right)^i \tag{20}$$

In this series above converges from $\left|\frac{y}{S^\alpha}\right| < 1$, hence,

$$\mathcal{L}\left(t^{\gamma-1} G_{\alpha,\gamma}(yt^\alpha)\right) = \frac{S^{-\gamma}}{1 - \frac{y}{S^\alpha}} = \left[\frac{S^{-(\gamma-\alpha)}}{S^\alpha - y}\right]. \tag{21}$$

4.1. Laplace transform of fractional derivatives for integer order

If f is of integer order, and f is continuous and f_0 is piecewise continuous on all interval [20].

$0 \leq \ t \leq b$: Then:

$$\mathcal{L}(f'(t)) = S\mathcal{L}(f(t)) - f(0)$$

Applying the theorem multiple times yields

$$\mathcal{L}(f''(t)) = S^2\mathcal{L}(f(t)) - Sf(0) - f'(0)$$

$$\mathcal{L}(f'''(t)) = S^3\mathcal{L}(f(t)) - S^2f(0) - f'(0) - f''(0)$$

$$\vdots$$

$$\mathcal{L}\left(f^{(n)}(t)\right) = S^n\mathcal{L}(f(t)) - S^{n-1}f(0) - S^{n-2}f'(0) - S^{n-3}f''(0) - \cdots - S^2f^{(n-3)}(0) - Sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{22}$$

Significantly, we say that the Laplace transform, when applied to differential equations, will change the derivatives into algebraic expressions in terms of s and the dependent variable t. Thus, the Laplace transform can convert a differential equation into an algebraic equation.

4.2. Laplace Transform of Fractional Differential Operators

Definition 4.2 Caputo Fractional Derivative

Assume that the function $f \in \mathbb{C}^n[a,b],\ \alpha \geq 0$ and $n - 1 < a \leq n$. Then we have

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \frac{\partial^n f(x)}{\partial x^n}\, dx,\ \ a \leq t < b$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(x)}{(t-x)^{\alpha+1-n}}\, dx. \tag{23}$$

Definition 4.3. The α^{th} order Riemann-Liouville derivative of function is.

$$D^\alpha_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_a^t \frac{f(x)\, dx}{(t-x)^{1-\alpha}}, \tag{24}$$

and the integral

$$I_{\alpha,t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} \frac{f(x)\,dx}{(t-x)^{1-\alpha}} \tag{25}$$

Lemma 4.1. The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be obtained in the form:

$$\mathcal{L}\big(I^n f(t)\big) = \frac{F(s)}{S^{\alpha}} \tag{26}$$

Where I^n is the α integral.

Proof. The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is get:

$$\mathcal{L}\big(I^n f(t)\big) = \mathcal{L}\left(\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-x)^{\alpha-1} f(x) dx\right) = \frac{1}{\Gamma(\alpha)} F(s) G(s)$$

Where is:

$$G(s) = \mathcal{L}(t^{\alpha-1}) = \frac{\Gamma(\alpha)}{S^{\alpha}}$$

And hence

$$\mathcal{L}\big(I^n f(t)\big) = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{S^{\alpha}} F(S) = \frac{F(s)}{S^{\alpha}}.$$

Lemma 4.2. The Laplace transform of Caputo fractional derivative for

$m-1 < \alpha \le m,\ m \in \mathbb{N}$, can be obtained in the form of [21, 22]:

$$\mathcal{L}\big(D_t^{\alpha} f(t)\big) = \frac{t^m f(t) - t^{m-1} f(0) - t^{m-2} f'(0) - \cdots - f^{(m-1)}(0)}{t^{m-\alpha}} \tag{27}$$

Proof. The Laplace transform of Caputo fractional derivative of order $\alpha > 0$ is:

$$\mathcal{L}\big(D_t^{\alpha} f(t)\big) = \mathcal{L}\left(I^{m-\alpha} f^{(m)}(t)\right) = \frac{\mathcal{L}\left(f^{(m)}(t)\right)}{S^{m-\alpha}} \tag{28}$$

We are ready to see how the Laplace transform can be used in differentiation equations.

4.3. Solving differential equation problems using the method of Laplace transform:

To solve a linear differential equation using Laplace transforms, there are only 3 basics steps:

1. Take the Laplace transforms of both sides of an equation.
2. Simplify algebraically the result to solve for $\mathcal{L}\big(f(t)\big) = F(s)$ in terms of s.
3. Find the inverse transform of $F(s)$. This inverse transform, $f(t)$, is the solution of the given differential equation. [20,21].

Example 4.1. We Consider the following differential equation:

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2, \qquad y'(0) = 1 \end{cases}$$

We transform both sides.

$$\mathcal{L}(y'')(s) + 5\mathcal{L}(y')(s) + 6\mathcal{L}(y)(s) = 0$$

From the equations (22), (28) to find $F(s) = \mathcal{L}(y)$

$$s^2\mathcal{L}(y)(s) - 2s - 1 + 5(s\mathcal{L}(y)(s) - 2) + 6\mathcal{L}(y)(s) = 0$$

Find the value

$$\mathcal{L}^{-1}\left(\frac{2s+11}{s^2+5s+6}\right)(t)$$

$$\frac{2s+11}{s^2+5s+6} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{s(A+B) + (2A+3B)}{(s^2+5s+6)}$$

Divide the equation using partial fractions.

$$2s = s(A+B) + 2A + 3B$$

$$\begin{cases} A + B = 2 \\ 2A + 3B = 11 \end{cases}$$

$$2A + 3B - (2A + 2B) = 11 - 2 \times 2 \Longrightarrow B = 7$$

And $A = 2 - B = 2 - 7 = -5$

$$\frac{2s+11}{s^2+5s+6}=\frac{-5}{s+3}+\frac{7}{s+2}$$

$$\mathcal{L}^{-1}\left(\frac{2s+11}{s^2+5s+6}\right)(t)=\mathcal{L}^{-1}\left(\frac{-5}{s+3}+\frac{7}{s+2}\right)(t)$$

$$-5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)(t)=-5e^{-3t}+7e^{-2t}$$

$$\mathcal{L}(y'')(s)=s^2\mathcal{L}(y)(s)-sy(0)-y'(0)=s^2\mathcal{L}(y)(s)-2s-1$$

$$\mathcal{L}(y')(s)=s\mathcal{L}(y)(s)-y(0)=s\mathcal{L}(y)(s)-2$$

We apply the Laplace transform to the differential equation.

$$\mathcal{L}(y'')(s)+5\mathcal{L}(y')(s)+6\mathcal{L}(y)(s)=0$$

$$s^2\mathcal{L}(y)(s)-2s-1+5(s\mathcal{L}(y)(s)-2)+6\mathcal{L}(y)(s)=0$$

Then,

$$(s^2+5s+6)\mathcal{L}(y)(s)=2s+11$$

So

$$\mathcal{L}(y)(s)=\frac{2s+11}{s^2+5s+6}$$

Example 4.2. We Consider the following partial differential equation:

$$\begin{cases}\dfrac{\partial u}{\partial x}=2\dfrac{\partial u}{\partial t}+u & x\geq 0, \qquad t\geq 0\\ u(x,0)=6e^{-3x}\end{cases}$$

Given $u(x,t)$ is boundary function for all $x\geq 0$, and $t\geq 0$

We apply the Laplace transform to the partial differential equation.

$$\mathcal{L}\left(\frac{\partial u}{\partial x}\right)=2\mathcal{L}\left(\frac{\partial u}{\partial t}\right)+\mathcal{L}(u)$$

Assume $\mathcal{L}(u)=V(x,s)$

Then,

$$\frac{d}{dx}\big(V(x,s)\big) = 2[sV(x,s) - u(x,0)] + V(x,s)$$

$$\Longrightarrow \frac{d}{dx}\big(V(x,s)\big) = 2sV(x,s) - 12e^{-3x} + V(x,s)$$

$$\Longrightarrow \frac{d}{dx}\big(V(x,s)\big) - (2s+1)V(x,s) = -12e^{-3x}$$

The last equation is an ordinary differential equation.

Example 4.3. We Consider the following differential equation:

$$\begin{cases} y' + 2y = 2te^{-2t} \\ y(0) = -3 \end{cases}$$

We transform both sides.

$$\mathcal{L}(y') + \mathcal{L}(2y) = \mathcal{L}(4e^{-2t})$$

$$s\mathcal{L}(y) - y(0) + 2\mathcal{L}(y) = \frac{2}{(s+2)^2}$$

To find $F(s) = \mathcal{L}(y)$

$$s\mathcal{L}(y) - (-3) + 2\mathcal{L}(y) = \frac{4}{(s+2)^2}$$

$$\mathcal{L}(y)(s+2) + 3 = \frac{4}{(s+2)^4}$$

$$\mathcal{L}(y)(s+2) = \frac{2}{(s+4)^2} - 3$$

$$\mathcal{L}(y) = \frac{4}{(s+2)^3} - \frac{3}{(s+2)} = \frac{4 - 3(s+2)^2}{(s+2)^3} = \frac{-3s^2 - 12s - 8}{(s+2)^3}$$

Divide the equation using partial fractions.

$$\mathcal{L}(y) = \frac{-3s^2 - 12s - 8}{(s+2)^3} = \frac{A}{(s+2)^3} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)} = \frac{A + B(s+2) + C(s+2)^2}{(s+2)^3}$$

$$\frac{-3s^2-12s-8}{(s+2)^3}=\frac{A+Bs+2B+Cs^2+4Cs+4C}{(s+2)^3}$$

$$\frac{-3s^2-12s-8}{(s+2)^3}=\frac{Cs^2+(B+4C)s+(A+2B+4C)}{(s+2)^3}$$

By equating the comparison for both fractions, we obtain:

$$C=-3,\quad (B+4C)=-12,\quad (A+2B+4C)=-8$$

Solving the above system, we obtain

$$C=-3,\qquad A=4,\qquad B=0$$

Now, by substituting the values in the expression of $\mathcal{L}(y)$, we obtain.

$$\mathcal{L}(y)=\frac{-3s^2-12s-8}{(s+2)^3}=\frac{4}{(s+2)^3}-\frac{3}{(s+2)}\Longrightarrow y(t)=4\mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right)-3\mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right)$$

And hence

$$y(t)=4t^2e^{-2t}-3e^{-2t}$$

In the next section, we will discuss how to solve differential equation problems for nonlinear fractions order for The Volterra Integral Equation.

5. The Volterra Integral Equation

This method introduced the series which converges to the solution of an initial-value problem. For the initial-value problem with the Riemann-Liouville derivative (24) appropriate sequence can be calculated in the following way. [10,15]:

$$y_0(t)=\sum_{s=1}^{n}\frac{b_s}{\Gamma(\alpha-S+1)}\,(t-a)^{\alpha-S} \tag{29}$$

$$y_i(t)=y_0(t)+\frac{1}{\Gamma(\alpha)}\int_a^t (t-\mu)^{\alpha-S} f\big(\mu,y_i(\mu)\big)d\mu \tag{30}$$

Where n is the number of initial conditions, $i=1,2,3,\ldots$ and $f\big(\mu,y_i(\mu)\big)$ is the right-hand side of the equation. Hence the solution is:

$$y(t) = \lim_{i\to\infty} y_i(t) \tag{31}$$

This method can be easily applied to nonlinear equations as well. And can Getting a formula in general may be a problem with a specification convergence period. And this method gives the solution in closed form to the linear binomial equation with constant coefficients and even to the equation:

$$D_a^{\alpha} y(t) - \beta(t-a)^{\lambda} y(t) = 0 \Rightarrow D_a^{\alpha-S} y(a) = b_S \tag{32}$$

where b_S, β are real constants, $S = 1, \dots, m$ and $\lambda > -\alpha$. Because we already know the solution of linear two-term equations with constant coefficients, we will solve the second problem now. Assume that without the proof that the problem (32) satisfies all necessary assumptions to this method [10,22].

Example 5.1. Solve the initial-value problem (29) with the Riemann-Liouville fractional derivative, $n = -[-\alpha]$.

Applying the formulas (30) and (31) we get the expressions.

$$y_0(t) = \sum_{S=1}^{n} \frac{b_s}{\Gamma(\alpha - S + 1)} (t-a)^{\alpha - S}$$

$$y_i(t) = y_0(t) + \frac{\beta}{\Gamma(\alpha)} \int_a^t (t-\mu)^{\alpha - S} (\mu - a)^{\lambda} y_{i-1}(\mu) d\mu$$

We compute terms $y_1(t), y_2(t)$ and see what happens.

$$y_1(t) = y_0(t) + \frac{\beta}{\Gamma(\alpha)} \int_a^t (t-\mu)^{\alpha - S} \sum_{S=1}^{n} \frac{b_s}{\Gamma(\alpha - S + 1)} (\mu - a)^{\alpha - \lambda - S}$$

$$= y_0(t) + \beta \sum_{S=1}^{n} \frac{b_s}{\Gamma(\alpha - S + 1)} D_a^{-\alpha}\left((t-a)^{\alpha - \lambda - S}\right)$$

$$y_0(t) + \beta \sum_{s=1}^{n} \frac{b_s (t-a)^{2\alpha + \lambda - s} \Gamma(\alpha + \lambda - S + 1)}{\Gamma(\alpha - S + 1)\ \Gamma(2\alpha + \lambda - S + 1)}$$

$$y_2(t) = y_0(t) \beta D_a^{-\alpha}\left((t-a)^{\lambda} y_1(t)\right) = y_1(t) + \beta^2 D_a^{-\alpha}\left((t-a)^{\lambda}\left(y_1(t) - y_0(t)\right)\right)$$

$$= y_1(t) + \beta \sum_{s=1}^{n} \frac{b_s(t-a)^{3\alpha+2\lambda-s}\Gamma(\alpha+\lambda-S+1)\Gamma(2\alpha+2\lambda-S+1)}{\Gamma(\alpha-S+1)\ \Gamma(2\alpha+\lambda-S+1)\Gamma(3\alpha+2\lambda-S+1)}$$

This can be proved in general by our:

$$y_i(t) = \sum_{S=1}^{n} \frac{b_s(t-a)^{\alpha-s}}{\Gamma(\alpha-S+1)} \left[1 + \sum_{i=1}^{j}\left(\prod_{\gamma=1}^{j} A_k \frac{\Gamma(\gamma(\alpha+\lambda)-S+1)}{\Gamma(\gamma(\alpha+\lambda)+\alpha-S+1)}\right)\left(\beta(t-a)^{\alpha+\lambda}\right)^i\right]$$

When we shift the index in the product and consider $m \to \infty$, we obtain the solution of this homogeneous equation containing the generalized function [23,24]:

$$y_n(t) = \sum_{S=1}^{n} \frac{b_s(t-a)^{\alpha-s}}{\Gamma(\alpha-S+1)} R_{\alpha,1+\frac{\lambda}{\alpha},1+\frac{\lambda-S}{\alpha}} \left(\beta(t-a)^{\alpha+\lambda}\right) \qquad (33)$$

Example 5.2. Solve the equation in the initial-value problem (29) with the Caputo fractional derivative and with initial conditions$y_s(a) = b_s$ for $S = 0,\dots,n-1$.
Here we solve the linear initial-value problem which we discussed generally for sequential derivative before. If we look at the Caputo derivative as its special case, implies the following procedure. Again, by application of (30) and (31) with:

$$y_0(t) = \sum_{s=1}^{n-1} \frac{b_s}{s!}(t-a)^s \qquad (34)$$

By following the same steps above, we obtain the expression for the i^{th} term:

$$y_i(t) = \sum_{s=1}^{n-1} \frac{b_s}{s!}(t-a)^s \left[1 + \sum_{i=1}^{j}\left(\prod_{\gamma=1}^{j} A_k \frac{\Gamma(\gamma(\alpha+\lambda)-S+1)}{\Gamma(\gamma(\alpha+\lambda)+\alpha-S+1)}\right)\left(\beta(t-a)^{\alpha+\lambda}\right)^i\right] \qquad (35)$$

Then we are using a limit and shift of the index we get the solution:

$$y_i(t) = \sum_{s=1}^{n-1} \frac{b_s}{s!}(t-a)^s R_{\alpha,1+\frac{\lambda}{\alpha},\frac{\lambda+s}{\alpha}}\left(\beta(t-\alpha)^{\alpha+\lambda}\right) \qquad (36)$$

We derived the solution of the homogeneous equation (34) with appropriate initial conditions in the Riemann-Liouville and the Caputo senses. It can be proven that the functions in th e sums which form both solutions, are independent [5]. We saw that due to the linearity it is not difficult to obtain the formula for the i^{th} term of the series and then to pass to the limit. Generally, the situation is not so simple.

$$\lim_{i\to n}\sum_{i=1}^{n-1}\frac{b_i}{i!}(t-a)^i$$

6. The Caputo Fractional Differential

Definition 6.1 (Caputo Fractional Derivative)

Assume the function $f\in\mathbb{C}^n[a,b], a>0$ and $n-1<\alpha\le n$, then,

$$\boldsymbol{D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-x)^{n-\alpha-1}\frac{\partial^n f(x)}{\partial x^n}dx=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}dx,\quad a\le t\le b,}\quad (37)$$

The benefit of using the Caputo definition is that it does not only allow for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to 0 [11,24].

Theorem 6.1. Fundamental Theorem of Calculus (FTC)

Let $f(x)$ be a continuous real-valued function defined on a closed interval $[a,b]$ and Let f be a real-valued function on a closed interval $[a,b]$ and F an antiderivative of f in $[a,b]$.

$$F(x)=\int_a^x f(x)\,dx,\qquad \forall\, x\in[a,b]. \qquad (38)$$

Then, $F(x)$ is uniformly continuous on $[a,b]$ differentiable on the open interval (a,b), and

$$F'(x)=f(x),\qquad \forall x\in(a,b) \qquad (39)$$

If f is Riemann integrable on, $[a,b]$ then.

$$\int_a^b f(x)\,dx=F(b)-F(a) \qquad (40)$$

$f(x)$ is n^{th} differentiable on $[a,b]$ then it is continuous since $f \in Cn[a,b]$ and $(t-x)_{n-\alpha-1}$ is continuous on the interval $[0,t)$ Since

$f^{(n)}(x)\ is\ bounded\ on\ [a,b], \quad and\ -1 < n-\alpha-1 \le 0,$ then

$$\frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}$$

Is integrable over $[0,t]$, where $a \le t \le b$. Thus, by FTC, $\frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}$ is differentiable and then it is continuous [5,25].

Example 6.1 Find the second derivative of $f(x) = x^3$ using Caputo definition.
From equation (37), we have.

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \frac{\partial^n f(x)}{\partial x^n} dx, \qquad 0 \le t < b$$

If $n = 3, \quad \alpha = 2$ then,

$$D_t^2 f(t) = \frac{1}{\Gamma(3-2)} \int_0^t (t-x)^{3-2-1} \frac{\partial^3 f(x)}{\partial x^3} dx,$$

For $f(x) = x^3 \Longrightarrow f'(x) = 3x^2, \quad f''(x) = 6x,\ and\ f'''(x) = 6$

Then,

$$D_t^2 f(t) = \frac{1}{\Gamma(1)} \int_0^t (t-x)^0\ 6\, dx, = \int_0^t 6\ \ dx = 6t$$

Note that $\Gamma(1) = 1$

Example 6.2. Find the half derivative of $f(x) = x^3$ using Caputo definition.
so $n = 3$ and $\alpha = 1/2$,

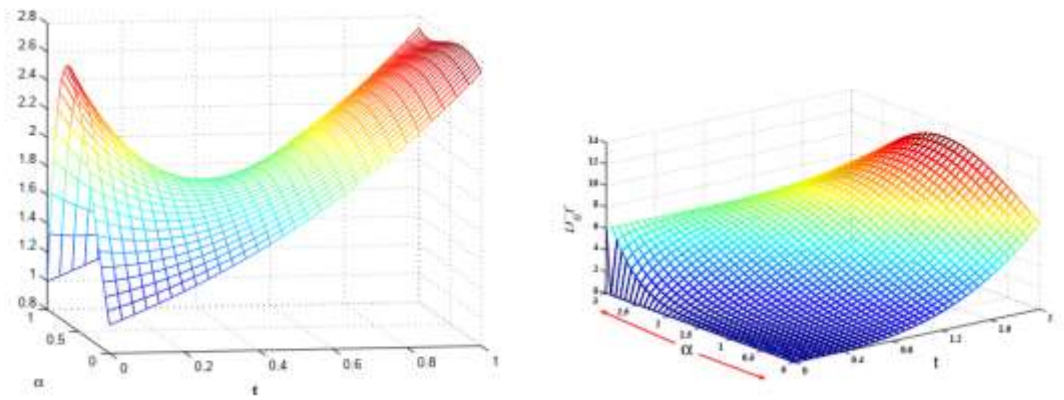
$$D_t^{\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(3-\frac{1}{2}\right)} \int_0^t (t-x)^{3-\frac{1}{2}-1} \frac{d^3 f(x)}{dx^3} dx,$$

Using the previous example, we have $f'''(x) = 6$ then, we have,

$$D_t^{\frac{1}{2}}\ f(t) = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}}\ 6\ dx = D_t^{\frac{1}{2}} f(t) = \frac{6}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}}\ dx.$$

Using the properties of Gamma function, the integral becomes formula,

$$D_t^{\frac{1}{2}}\ f(t) = \frac{6\,.\,2}{5.\Gamma\left(2+\frac{1}{2}\right)}\ (t-x)^{\frac{3}{2}+1}\Big|_0^t\ = \frac{12}{5.\frac{3}{4}\Gamma\left(\frac{1}{2}\right)}(-t)^{\frac{5}{2}} = \frac{8}{15\sqrt{\pi}}.(-t)^{\frac{5}{2}}$$



(1) $D_t^{\alpha}(f(t) = t^3),\quad 0 \le \alpha \le 2$ (2) $D_t^{\alpha}(f(t) = t^3),\ 0 \le \alpha \le \frac{1}{2}$

Figures 6.1 (1)-(2)

Remark 6.1. The fractional derivatives and integrals of function $f(x) = x^3$ in example 6.1 and 6.2 plotted in Figures 6.1 (1)-(2) are computed by applying definition 6.1 and Theorem 6.1. The fraction derivatives and integrals of $f(t) = t^3$ are evaluated by the application of Lemma 4.2. The fractional derivatives and integrals of trigonometric and hyperbolic functions can be evaluated using the relation between Volterra Integral function and generalized trigonometric functions (35), generalized hyperbolic functions (37). But the numerical evaluation of the Volterra Integral functions is itself difficult. We have used a much simpler method based on the Haar wavelets, to evaluate the fractional integrals of some functions of Caputo Fractional Differential. For the classical cases. $\alpha = 2, \frac{1}{2}$. The obtained results by the Haar wavelets are in good agreement with the exact values. For $f(t) = t^3$ and $\alpha = 2, \frac{1}{2}$, the maximum absolute error is 6.5×10^{-4} × 10-4 and $\frac{8}{15\sqrt{\pi}} \times 10^{-5}$ respectively.

7. Conclusions

The classical tools from functional analysis operator theory, on existence to boundary value problems for nonlinear fractional differential equations, with the. Laplacian Transform Interpretation and the Caputo fractional derivatives, Volterra Integral Equation, Caputo Fractional Differential is developed, we established sufficient conditions for existence results for different classes of nonlinear boundary value problems involving fractional derivatives, subject to integral boundary conditions. Several existence results for positive and multiple positive solutions to different ways of boundary value problems for fractional differential equations are obtained. For the value problem (1), the existence of at least one positive solution is guaranteed in a specially constructed cone in the Laplace transform of Riemann-Liouville fractional integral operator. It is observed that functions (22) and (23) for the value problem (28), satisfy some interesting the Volterra Integral Equation, Caputo Fractional Differential and useful properties and they are related to each other. This helps us to construct a cone in the partial differential equation model. Then we established existence results for positive solutions in this cone.

8. References

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